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A model for magneto-elastic behaviour

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Summary.   In this paper, a coupled magnetoelastic model for isotropic ferromagnetic materials
is presented. The constitutive equations are written on the basis of the total energy in which
the right Cauchy-Green strain tensor and the Lagrangian form of the magnetic field strength
are used as the basic state variables. It is also applied to ferromagnetic electric steel for which
the material parameters are calibrated.

Introduction

In this paper a finite strain model for isotropic magnetostriction is developed. The model
is based on the formulation introduced by Dorfmann and Ogden [3, 4] utilizing the concept
of "total energy function". Since further development towards anisotropic behaviour is
looked for, the formulation is given by using the Lagrangian description.

Constitutive equations

In magnetoelastostatics the three basic magnetic variables are the magnetic field $\boldsymbol{H}$, the
magnetic induction $\boldsymbol{B}$ and the magnetization $\boldsymbol{M}$. The fields $\boldsymbol{H}$ and $\boldsymbol{B}$ are considered as
the primary fields and $\boldsymbol{M}$ only as an auxiliary field [5], which can be defined in terms of
$\boldsymbol{H}$ and $\boldsymbol{B}$.

In electromagnetics it is customary to work with the Eulerian frame, so the field $\boldsymbol{H}$ and
$\boldsymbol{B}$ are related to the current configuration. Since our future goal is to model anisotropic
behaviour where the material orientation is important, a material description of motion
is chosen. The Lagrangian forms of the primary magnetic fields are

$$\boldsymbol{H}_{\mathrm{L}} \equiv \boldsymbol{F}^T \boldsymbol{H}, \qquad \text{and} \qquad \boldsymbol{B}_{\mathrm{L}} \equiv J\boldsymbol{F}^{-1}\boldsymbol{B}, \tag{1}$$

where $\boldsymbol{F}$ is the deformation gradient and $J = \det \boldsymbol{F}$. For further details see [3, 4].

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Denoting the complementary form of the total energy function as $\Omega^*(\boldsymbol{F}, \boldsymbol{H}_\mathrm{L})$, and using the standard Coleman-Noll procedure, the total stress $\boldsymbol{\tau}$ and the magnetic induction $\boldsymbol{B}$ can be obtained from equations

$$\boldsymbol{\tau} = J^{-1}\boldsymbol{F}\frac{\partial\Omega^*}{\partial\boldsymbol{F}}, \qquad \boldsymbol{B} = -J^{-1}\boldsymbol{F}\frac{\partial\Omega^*}{\partial\boldsymbol{H}_\mathrm{L}}. \tag{2}$$

The function $\Omega^*(\boldsymbol{F}, \boldsymbol{H}_\mathrm{L})$ is a partial Legendre transform of the total energy function $\Omega(\boldsymbol{F}, \boldsymbol{B}_\mathrm{L})$, i.e.

$$\Omega^*(\boldsymbol{F}, \boldsymbol{H}_\mathrm{L}) = \Omega(\boldsymbol{F}, \boldsymbol{B}_\mathrm{L}) - \boldsymbol{H}_\mathrm{L}\cdot\boldsymbol{B}_\mathrm{L}. \tag{3}$$

The total energy function Ω is related to the Helmholtz free energy per unit mass ψ as

$$\Omega \equiv \rho_0\Phi + \tfrac{1}{2}\mu_0^{-1}J\boldsymbol{B}\cdot\boldsymbol{B}, \qquad \text{where} \qquad \Phi(\boldsymbol{F}, \boldsymbol{B}_\mathrm{L}) \equiv \psi(\boldsymbol{F}, J^{-1}\boldsymbol{F}\boldsymbol{B}_\mathrm{L}) \tag{4}$$

in which ρ_0, μ_0 are the density in the reference configuration and the magnetic permeability in vacuum, respectively.

To model isotropic magnetostriction, the energy is assumed to depend on the right Cauchy-Green deformation tensor $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ and the Lagrangian magnetic field $\boldsymbol{H}_\mathrm{L}$. Integrity basis of a scalar function depending of a symmetric second order tensor and a vector consist of the following six invariants [6]:

$$I_1 = \operatorname{tr}\boldsymbol{C}, \ I_2 = \tfrac{1}{2}[(\operatorname{tr}\boldsymbol{C})^2 - \operatorname{tr}\boldsymbol{C}^2], \ I_3 = \det\boldsymbol{C}, \ I_4 = \boldsymbol{H}_\mathrm{L}\cdot\boldsymbol{H}_\mathrm{L},$$
$$I_5 = \boldsymbol{H}_\mathrm{L}\cdot\boldsymbol{C}\boldsymbol{H}_\mathrm{L}, \ I_6 = \boldsymbol{H}_\mathrm{L}\cdot\boldsymbol{C}^2\boldsymbol{H}_\mathrm{L}. \tag{5}$$

From equations (2) expressions for the total stress tensor $\boldsymbol{\tau}$ and the magnetic induction vector $\boldsymbol{B}$ are found as

$$\boldsymbol{\tau} = J^{-1}\boldsymbol{F}\sum_{k=1}^6\frac{\partial\Omega^*}{\partial I_k}\frac{\partial I_k}{\partial\boldsymbol{F}}, \qquad \boldsymbol{B} = -J^{-1}\boldsymbol{F}\sum_{k=1}^6\frac{\partial\Omega^*}{\partial I_k}\frac{\partial I_k}{\partial\boldsymbol{H}_\mathrm{L}}. \tag{6}$$

Evaluation of the derivatives $\partial\Omega^*/\partial\boldsymbol{F}$ and $\partial\Omega^*/\partial\boldsymbol{H}_\mathrm{L}$ gives

$$\boldsymbol{\tau} = J^{-1}[2\boldsymbol{b}\Omega_1^* + 2(I_1\boldsymbol{b} - \boldsymbol{b}^2)\Omega_2^* + 2I_3\boldsymbol{I}\Omega_3^* + 2\boldsymbol{b}\boldsymbol{H}\otimes\boldsymbol{b}\boldsymbol{H}\Omega_5^*$$
$$+ 2(\boldsymbol{b}\boldsymbol{H}\otimes\boldsymbol{b}^2\boldsymbol{H} + \boldsymbol{b}^2\boldsymbol{H}\otimes\boldsymbol{b}\boldsymbol{H})\Omega_6^*], \tag{7}$$
$$\boldsymbol{B} = -J^{-1}(2\boldsymbol{b}\boldsymbol{H}\Omega_4^* + 2\boldsymbol{b}^2\boldsymbol{H}\Omega_5^* + 2\boldsymbol{b}^3\boldsymbol{H}\Omega_6^*), \tag{8}$$

where $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T$ is the left Cauchy-Green deformation tensor, the notation Ω_i^* denotes the derivative $\Omega_i^* = \partial\Omega^*/\partial I_i$ and $\otimes$ is the standard tensor product.

The chosen total energy functional is additively decomposed to a purely elastic part

$$\Omega^{*\mathrm{e}} = \frac{1}{2}K\left[\frac{1}{2}(J^2 - 1) - \ln J\right] + \frac{1}{2}G(\operatorname{tr}\bar{\boldsymbol{C}} - 3) \tag{9}$$

and a coupled magneto-elastic part

$$\Omega^{*\mathrm{m}} = -\tfrac{1}{2}\mu_0(1 - \alpha_5 - \alpha_6)I_4 - \alpha_4\mu_0 h_0\ln\left[\cosh(f(x)\sqrt{I_4}/h_0)\right]$$
$$-\tfrac{1}{2}\alpha_5\mu_0 h_0^2(I_5/h_0^2) - \tfrac{1}{2}\alpha_6\mu_0 h_0^2\left\{1 + \beta_1\left[1 - \exp(-\beta_2 I_4/h_0^2)\right]\right\}(I_6/h_0^2), \tag{10}$$

where the isochoric part of the right Cauchy-Green deformation tensor is denoted as $\bar{\boldsymbol{C}} = J^{-2/3}\boldsymbol{C}$ and K, G are the bulk and shear moduli, respectively. The function $f(x)$, where

Table 1. Coefficients of the f-function.

k_f	k_s	$10^4 \cdot d$	s^-	s^+	f_∞^-	f_∞^+	$10^4 \cdot x_m$	f_m
5378.499	4431.884	1.75633	1.6775	0.531416	0.22914	1.00759	1.089	2.4
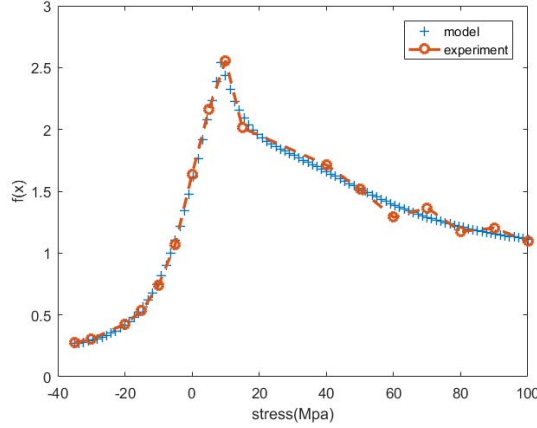
Figure 1. Measured and model forms of the f-function.

$x = \tilde{I}_5 = \boldsymbol{H}_\mathrm{L} \cdot (\bar{\boldsymbol{C}} - \boldsymbol{I}) \cdot \boldsymbol{H}_\mathrm{L}/I_4$ is important in modelling the deformation dependency, i.e. the stress dependency of the initial magnetic permeability. For electric steel the following form is adopted based on experimental results

$$f(x) = f_\infty(x) + (f_\mathrm{m} - f_\infty(x)) \exp\left\{-\sqrt{[(s(x)x - x_\mathrm{m})/d]^2}\right\}, \tag{11}$$

and the infinity f_∞ and slope function s are of the form

$$f_\infty(x) = \frac{1}{2}\left\{f_\infty^- + f_\infty^+ + (f_\infty^+ - f_\infty^-)\tanh\left[k_f(x - x_\mathrm{m})\right]\right\}, \tag{12}$$

$$s(x) = \frac{1}{2}\left\{s^- + s^+ + (s^+ - s^-)\tanh\left[k_s(x - x_\mathrm{m})\right]\right\}. \tag{13}$$

Note that the function f is insensitive to volumetric deformation. The behaviour of the initial magnetic permeability as a function of applied stress shows a peak and saturation both in tension and compression, therefore the rather complicated expression (11) contains nine parameters: $f_\mathrm{m}, x_\mathrm{m}, d, f_\infty^-, f_\infty^+, k_f, s^+, s^-$ and k_s, which are estimated by standard least-squares fitting. The values obtained for the electric steel of grade Mxxx-50A[2] are shown in Table 1, see also Fig. 1. It should be noticed that the form of the function f is material dependent.

The model parameters $(K, G, h_0, \alpha_4, \alpha_5, \alpha_6, \beta_1,$ and $\beta_2)$ will be estimated from the experiments. It can be seen that depending on the signs and relative magnitudes of the coefficients α_5 and α_6, the following behavior can be obtained in the uniaxial case:

1. $\alpha_5 < 0, \alpha_6 = 0$ results in negative magnetostriction with saturation,
2. $\alpha_5 > 0, \alpha_6 = 0$ results in positive with H monotonously increasing magnetostriction,

[2]Unfortunately the loss value of the studied material is not known.

3. $\alpha_5 = 0, \alpha_6 > 0, \beta_1 = 0$ results in positive with H monotonously increasing magnetostriction,

4. $\alpha_5 > 0, \alpha_6 < 0, \beta_1 = 0$ results in negative saturating magnetostriction,

5. $\alpha_5 > 0, \alpha_6 < 0, \beta_1 \neq 0$, but $|\alpha_6| < \frac{1}{2}\alpha_5$ results in positive saturating magnetostriction,

6. $\alpha_5 > 0, \alpha_6 < 0, \beta_1 \neq 0$, but $|\alpha_6| > \frac{1}{2}\alpha_5$ results in initially positive magnetostriction which however saturates to a negative value.

Concluding remarks

Modelling magnetoelastic behaviour is challenging due to the complexity of the coupled magnetomechanical problem. A Lagrangian formulation is adopted due to future expandability to an anisotropic case. Future study is still needed to explore the capabilities of the model.

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