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A posteriori analysis of classical plate elements

Tom Gustafsson, Rolf Stenberg[1] and Juha Videman

Summary. We outline the results of our recent article on the a posteriori error analysis of C^1 finite elements for the classical Kirchhoff plate model with general boundary conditions. Numerical examples are given.

Key words: Kirchhoff plate model, C^1 elements, a posteriori error estimates.

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Introduction

The purpose of our work is to fill a gap in the literature. Surprisingly, the a posteriori error analysis for classical plate finite elements has so far only been given for the fully clamped case and a load in L^2, cf. [2]. In our recent work [1], we treated a combination of all common boundary conditions (clamped, simply supported and free). In addition, we considered the cases of point and line loads.

The Kirchhoff plate problem

We denote the deflection of the plate's midsurface by u, the curvature by $\boldsymbol{K}$ and the moment by $\boldsymbol{M}$, and we assume isotropic linear elasticity. Hence, it holds

$$\boldsymbol{M}(u) = \frac{d^3}{12}\mathbb{C}\,\boldsymbol{K}(u), \tag{1}$$

with

$$\mathbb{C}\,\boldsymbol{A} = \frac{E}{1+\nu}\big(\boldsymbol{A} + \frac{\nu}{1-\nu}(\operatorname{tr}\boldsymbol{A})\boldsymbol{I}\big), \quad \forall \boldsymbol{A} \in \mathbb{R}^{2\times 2}, \tag{2}$$

where d denotes the thickness of the plate. E and ν are the Young's modulus and Poisson ratio, respectively. The strain energy for an admissible deflection v is then $\frac{1}{2}a(v,v)$, with

$$a(w,v) = \int_\Omega \boldsymbol{M}(w) : \boldsymbol{K}(v)\,\mathrm{d}x = \int_\Omega \frac{d^3}{12}\mathbb{C}\,\boldsymbol{\varepsilon}(\nabla w) : \boldsymbol{\varepsilon}(\nabla v)\,\mathrm{d}x. \tag{3}$$

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The potential energy $l(v)$ stems from the loading, which we assume to consist of a distributed load $f \in L^2(\Omega)$, a load $g \in L^2(S)$ along the line $S \subset \Omega$, and of a point load F at the point x_0, so that

$$l(v) = \int_\Omega f v \,\mathrm{d}x + \int_S g v \,\mathrm{d}s + F v(x_0). \tag{4}$$

The total energy is thus $\frac{1}{2} a(v,v) - l(v)$, and its minimisation leads to the variational form: *find $u \in V$ such that*

$$a(u,v) = l(v) \quad \forall v \in V, \tag{5}$$

with

$$V = \{\, v \in H^2(\Omega) \,|\, v|_{\Gamma_\mathrm{c} \cup \Gamma_\mathrm{s}} = 0, \quad \frac{\partial u}{\partial n}|_{\Gamma_\mathrm{c}} = 0 \}. \tag{6}$$

We assume that the plate is clamped on the boundary part Γ_c, simply supported on Γ_s, and free on $\Gamma_\mathrm{f} = \partial\Omega \setminus (\Gamma_\mathrm{c} \cup \Gamma_\mathrm{s})$.

By the well-known integration by parts, we get the boundary value problem. To this end we have to recall the following quantities for a admissible displacement v; the normal shear force $Q_n(v)$, the normal and twisting moments $M_{nn}(v)$, $M_{ns}(v)$, and the effective shear force

$$V_n(v) = Q_n(v) + \frac{\partial M_{ns}(v)}{\partial s}. \tag{7}$$

With the constitutive relationship (2), an elimination yields the plate equation for the deflection u:

$$\mathcal{A}(u) := D\Delta^2 u = l, \tag{8}$$

where the so-called bending stiffness D is defined as

$$D = \frac{Ed^3}{12(1-\nu^2)}. \tag{9}$$

The boundary value problem is the following.

- In the domain we have the distributional *differential equation*

  $$\mathcal{A}(u) = l \quad \text{in } \Omega, \tag{10}$$

  where l is the distribution defined by (4).

- On the *clamped* part we have the conditions: $u = 0$ and $\frac{\partial u}{\partial n} = 0$ on Γ_c.

- On the *simply supported* part it holds: $u = 0$ and $M_{nn}(u) = 0$ on Γ_s.

- On the *free part* it holds: $M_{nn}(u) = 0$ and $V_n(u) = 0$ on Γ_f.

- At the *corners on the free part* we have the jump condition on the twisting moment

  $$[\![M_{ns}(u)(c)]\!] = 0 \quad \text{for all corners } c \in \Gamma_\mathrm{f}.$$

Here and below $[\![\cdot]\!]$ denotes the jump.

We consider conforming finite element methods: *find $u_h \in V_h \subset V$ such that*

$$a(u_h, v) = l(v) \quad \forall v \in V_h. \tag{11}$$

The finite element partitioning is denoted by $\mathcal{C}_h$. We assume that mesh is such that the point load is a vertex and the line load is along edges. The edges are divided into interior edges $\mathcal{E}_h^i$, edges on S, $\mathcal{E}_h^S$, edges on the free boundary $\mathcal{E}_h^f$, and edges on the simply supported boundary $\mathcal{E}_h^s$. The local error indicators are then the following.

- The residual on each element

$$h_K^2 \|\mathcal{A}(u_h) - f\|_{0,K}, \quad K \in \mathcal{C}_h.$$

- The jump residuals of the normal moment along interior edges

$$h_E^{1/2} \|[\![M_{nn}(u_h)]\!]\|_{0,E}, \quad E \in \mathcal{E}_h^i.$$

- The jump residuals in the effective shear force along interior edges

$$h_E^{3/2} \|[\![V_n(u_h)]\!] - g\|_{0,E}, \quad E \in \mathcal{E}_h^S, \qquad h_E^{3/2} \|[\![V_n(u_h)]\!]\|_{0,E}, \quad E \in \mathcal{E}_h^i \setminus \mathcal{E}_h^S.$$

- The normal moment along edges on the free and simply supported boundaries

$$h_E^{1/2} \|M_{nn}(u_h)\|_{0,E}, \quad E \in \mathcal{E}_h^f \cup \mathcal{E}_h^s.$$

- The effective shear force along edges on the free boundary

$$h_E^{3/2} \|V_n(u_h)\|_{0,E}, \quad E \in \mathcal{E}_h^f.$$

The error estimator is defined through

$$\begin{aligned} \eta^2 = &\sum_{K \in \mathcal{C}_h} h_K^4 \|\mathcal{A}(u_h) - f\|_{0,K}^2 + \sum_{E \in \mathcal{E}_h^S} h_E^3 \|[\![V_n(u_h)]\!] - g\|_{0,E}^2 + \sum_{E \in \mathcal{E}_h^i \setminus \mathcal{E}_h^S} h_E^3 \|[\![V_n(u_h)]\!]\|_{0,E}^2 \\ &+ \sum_{E \in \mathcal{E}_h^i} h_E \|[\![M_{nn}(u_h)]\!]\|_{0,E}^2 + \sum_{E \in \mathcal{E}_h^f} h_E^3 \|V_n(u_h)\|_{0,E}^2 + \sum_{E \in \mathcal{E}_h^f \cup \mathcal{E}_h^s} h_E \|M_{nn}(u_h)\|_{0,E}^2. \end{aligned} \tag{12}$$

Our a posteriori estimate is the following, where the energy norm is defined as $|||v||| = a(v,v)^{1/2}$.

Theorem 1 *There exists positive constants C_1, C_2, such that*

$$C_1 \eta \leq |||u - u_h||| \leq C_2 \eta. \tag{13}$$

Numerical examples

In the examples, we have used the Argyris triangle. In the figures, we give the meshes for the adaptive solution of a square plate with a point and line load, and for a L-shaped domain with a free boundary for the edges sharing the re-entrant corner and simply supported along the rest of the boundary.

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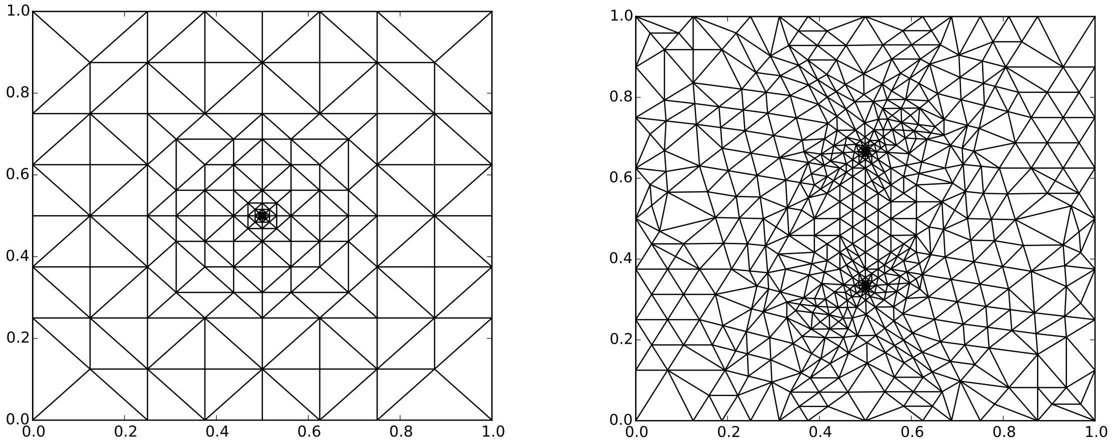

Figure 1. The adaptive meshes for the point and line loads.

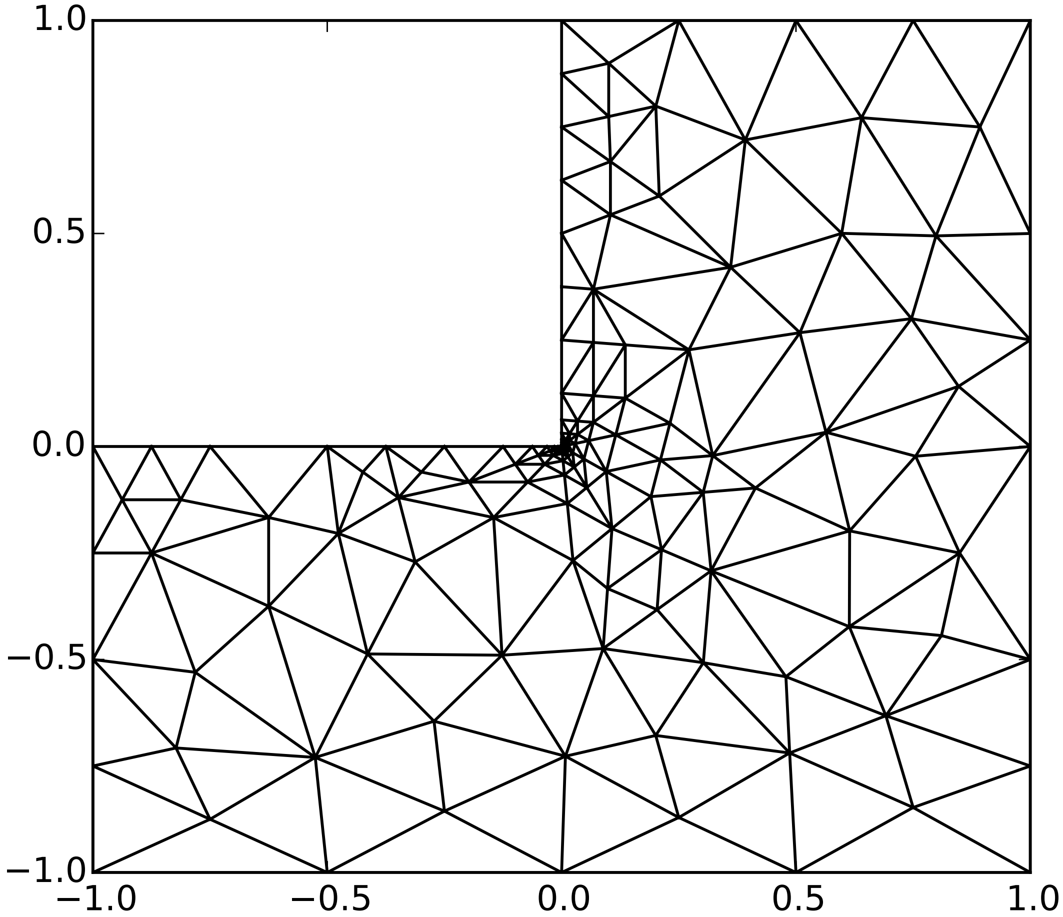

Figure 2. The adaptive mesh for the L-shaped domain.

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