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On the derivation of constant-coefficient partial differential equations for elastic shells

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Summary Here the problem of formulating a representative model problem of shell theory is considered. We study two ways to obtain a constant-coefficient expression for the strain energy density function of a linearly elastic shell. The first formulation has already been given in the context of the analysis of boundary layers in thin shells, while the other is introduced here. It appears that the essential difference between the formulations is that the constant-coefficient expressions for the strains given here depend on four geometric parameters instead of the two parameters of curvature needed by the earlier derivation. The source of this discrepancy is investigated and shown to be related to the properties of the metric tensors that are attainable by means of different parametrizations of a given surface.

Keywords: shell, line of curvature, reparametrization, surface, shallow shell

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Introduction

Utilizing a representative model problem is a standard technique to approach the mathematical analysis of partial differential equation models. Ideally, such model problem should capture all essential aspects of the cases of interest, while being as simple as possible to make the analysis easier, or even to allow for the derivation of analytical solutions of special cases. The formulation of an ideal model problem is not, however, a straightforward question in the case of shell equations characterizing the deformation of a curved three-dimensional body of a small thickness, as generally the pertinent equations are quite cumbersome. Moreover, the comparison of different versions of shell equations is complicated owing to the use of different approximations and various ways to represent the equations (as an example, one may choose to use either the Christoffel symbols or geometric Láme parameters to describe the necessary relations of differential geometry).

A special feature of the shell equations is that a parametrization of the shell midsurface has a crucial effect on the detailed expressions attained for the field equations, and possible complications caused by the choice of the parametrization are eventually felt by the analyst when the solution of a special case is attempted. The geometry representation thus affects the parametrized form of strains and possibly expressions for a material law. It is clear that here a clever surface parametrization can be very helpful. In particular, the orthogonality of a surface parametrization is seen to be desirable so that special simplicity may be attained (especially, the application of the material law to describe the strain energy density is then made easier). It should be noted that here an orthogonal parametrization is understood to mean a representation leading to orthogonal curvilinear coordinates in the surface.

We note that lines of curvature enable an orthogonal parametrization in special cases and that historically they have been applied commonly to treat special problems of shell theory with analytic methods. This technique is based on the differential geometric result that the minimal curvature and maximal curvature of a smooth surface are characterized by two orthogonal directions (i.e., the principal directions). This enables the generation of a line of curvature as an integral curve on the surface whose tangent vector is always aligned with a principal direction of one type. In general the definition of lines of curvature coordinates may however be problematic, since the definition of a net of orthogonal curvilinear coordinates suffers from an ambiguity at an umbilical point where the maximal and minimal curvatures are non-zero and the same. We note that such singular points are often isolated.

Now, if a parametrization by lines of curvature is possible, it is a candidate for a perfect choice for the formulation of model problem related to the analysis of thin shells. As already mentioned, a global parametrization of a general mid-surface by lines of curvature coordinates may not exist owing to the singularities. However, a local version may often be generated, so this technique is indeed useful to treat local analyses despite we may miss the global form of the surface representation in this way.

Further simplifications may also be attempted by seeking approximations which lead to formulating the shell problem in terms of a constant-coefficient system of partial differential equations. Such equations have traditionally been given in the context of shallow shell theory where the shell mid-surface is typically defined in the form $z = f(x, y)$. If $f(x, y)$ is replaced by a polynomial expression of order 2 in the variables x and y, constant coefficients may be obtained; cf., for example, [10].

Although the primary aim of the much later paper of Pitkäranta, Matache and Schwab [9] is not related to the formulation of shell models, their study nevertheless describes an interesting derivation of simplified models that are suitable for analysing shallow shells. This derivation is motivated by the desire to obtain a constant-coefficient system of partial differential equations over a two-dimensional domain, so that the geometric parameters need not be considered to depend on place. A detailed treatment of boundary layer effects related to the modelling of thin elastic shells is then given in [9] by using these equations. We note that the models considered bear a similarity to those of other studies [2, 6, 7] and [8], where a constant-coefficient approximation is also employed in the context of the analysis and design of finite element methods for shell problems.

In the study [9] the derivation of simplified shell models begins with the assumption that a global parametrization of the shell mid-surface by lines of curvature coordinates is available. Additional assumptions are then introduced so that the shell equations are finally expressed with constant coefficients by using a representation which gives only an approximation to lines of curvature coordinates. This process of approximation, however, appears to be proceeding in the reverse direction to what may occur in the practical modelling of surfaces. As obtaining a parametrization by lines of curvature coordinates is not straightforward in general, it is more common in practice that one has some initial parametrization of a surface and then one may proceed to seek for a more convenient reparametrization which gives an approximation to lines of curvature coordinates. Such a procedure has previously been considered by the present author in [3].

The aim of the present paper is to demonstrate the utility of the differential geometric results obtained in [3] by considering the derivation of simplified shell models. In particular, to pursue the reductions sought in [9], we consider the approximations which are seen to be necessary so as to obtain a constant-coefficient shell model. The direct use of the results already derived in [3] actually makes this procedure very straightforward. We also study connections between the present approach and the approximation procedure which is described in [9]. As we shall see that the two procedures lead to the strain expressions depending on different sets of constant parameters, it will be considered worthwhile to study the origin of this discrepancy.

To make a broader comparison with related studies, we note that quite similar simplifications of shell equations in the vicinity of a preassigned point P may also be obtained without confining to lines of curvature coordinates. In particular, the study [5] is focused on the case of locally Cartesian coordinates on a curved surface which are such that the associated components of the first fundamental form (called the metric tensor) have zero partial derivatives of first order at P. This implies that the Christoffel symbols vanish at P (whence they may also be considered relatively small in the immediate vicinity of P), and therefore simpler formulae for differentiation may be obtained since the first partial derivatives of tensor components can be used to evaluate the corresponding covariant derivatives at P. The development presented in [5] does not thus impose a restriction on the possible form of the second fundamental form, which is needed to describe the curvatures of the surface. The absence of such a constraint is also characteristic of the constant-coefficient formulations of the related studies [2, 6, 7] and [8]. Moreover, it is readily observed that the development given in [2, Section 2] particularly conforms to the definition of locally Cartesian coordinates.

On the other hand, the reparametrization method of [3] which we apply here requires that the components of both the second and first fundamental forms remain diagonal up to an approximation of optimal degree as we move away from P. As these prerequisites are different from that of [5], this does not in general lead to curvilinear coordinates which are locally Cartesian in the sense of the above definition. Indeed, we shall demonstrate that the Christoffel symbols evaluated at P may be nonzero and therefore the theory developed in [5] does not apply. This paper will arrive at the conclusion that the discrepancy of the constant-coefficient expressions of strains is intimately related to the properties of the metric tensors that are attainable by means of different parametrizations of a given surface.

Our plan is first to introduce some basic definitions which enable us to introduce fields which measure strain of a thin shell. After these preliminaries, we show how the differential geometric results obtained in [3] give the expressions for the strains over a local region in a straightforward manner. The connection of this derivation and the formulation of [9] is then studied. For brevity of exposition, we do this by considering in more detailed way only the expressions for the membrane strains. Finally, before making concluding remarks, a simple application is also given to illustrate that the four-parameter strain expressions cannot be simplified in practical cases, so they are in general believed to be the simplest consistent approximations when the surface parametrization is based on [3]. On the other hand, the two-parameter expressions can be reached at the cost of having greater errors in the diagonalization conditions of the fundamental forms as compared with errors that would have been attainable by the full use of the geometric parameters assumed to be known in [9]. It should be mentioned that in the following the standard summation convention of tensor analysis will be used; cf. [4].

Preliminaries

In order to introduce fields which measure strain of a shell body, we assume that its mid-surface $\mathcal{S} \subset \mathbb{E}^3$ is parametrized as

$$\omega \ni (y^1, y^2) \equiv \mathbf{y} \mapsto \boldsymbol{\varphi}(\mathbf{y}) \in \mathcal{S}, \tag{1}$$

where $\omega \subset \mathbb{R}^2$ is supposed to be associated with orthonormal basis vectors $\hat{\boldsymbol{e}}_\alpha$. Since we are restricting our attention to an orthogonal parametrization, the differentiation of this function as

$$\hat{\boldsymbol{a}}_\alpha(\mathbf{y}) = D\boldsymbol{\varphi}(\mathbf{y})[\hat{\boldsymbol{e}}_\alpha] = \partial_\alpha \boldsymbol{\varphi}(\mathbf{y}) \tag{2}$$

gives an orthogonal system of covariant basis vectors $\boldsymbol{a}_\alpha : \mathcal{S} \to \mathbb{R}^3$ via the alternate descriptions of $\hat{\boldsymbol{a}}_\alpha$ as

$$\boldsymbol{a}_\alpha(\mathbf{p}) \equiv (\hat{\boldsymbol{a}}_\alpha \circ \boldsymbol{\varphi}^{-1})(\mathbf{p}). \tag{3}$$

In order to obtain a system of normal coordinates, we also define

$$\hat{\boldsymbol{a}}_3 = \frac{\hat{\boldsymbol{a}}_1 \times \hat{\boldsymbol{a}}_2}{\sqrt{a}}, \tag{4}$$

with a denoting the determinant of the metric surface tensor defined by $A_{\alpha\beta} = \hat{\boldsymbol{a}}_\alpha \cdot \hat{\boldsymbol{a}}_\beta$. In addition, the second fundamental form and the Christoffel symbols are defined by

$$B_{\alpha\beta} = \hat{\boldsymbol{a}}_3 \cdot \partial_\alpha \hat{\boldsymbol{a}}_\beta = -\hat{\boldsymbol{a}}_\alpha \cdot \partial_\beta \hat{\boldsymbol{a}}_3 \tag{5}$$

and

$$\Gamma^\gamma_{\alpha\beta} = \partial_\beta \hat{\boldsymbol{a}}_\alpha \cdot \hat{\boldsymbol{a}}^\gamma, \tag{6}$$

where $\hat{\boldsymbol{a}}^\alpha$ are the contravariant basis vectors such that $\hat{\boldsymbol{a}}_i \cdot \hat{\boldsymbol{a}}^j = \delta_i^j$, with δ_i^j the Kronecker delta. In the case of a parametrization by lines of curvature coordinates the principal curvatures may now be expressed as (we note that these definitions depend on a choice of sign)

$$\frac{1}{R_1(\mathbf{y})} = -\frac{B_{11}(\mathbf{y})}{A_{11}(\mathbf{y})} = -B_1^1(\mathbf{y}) \quad \text{and} \quad \frac{1}{R_2(\mathbf{y})} = -\frac{B_{22}(\mathbf{y})}{A_{22}(\mathbf{y})} = -B_2^2(\mathbf{y}). \tag{7}$$

In the traditional shell theory the displacement field $\boldsymbol{u} : \mathcal{S} \times [-d/2, d/2] \to \mathbb{R}^3$ of the shell of thickness d is written in the form

$$\boldsymbol{u}(\boldsymbol{\varphi}(\mathbf{y}), y^3) \equiv \hat{\boldsymbol{u}}(\mathbf{y}, y^3) \equiv \boldsymbol{v}(\mathbf{y}) - y^3 \boldsymbol{\beta}(\mathbf{y}), \tag{8}$$

where $\boldsymbol{v} : \omega \to \mathbb{R}^3$ gives the mid-surface displacement and the field $\boldsymbol{\beta}$ related to rotations is assumed to be of the form $\boldsymbol{\beta} = \hat{\boldsymbol{a}}_3 \times \boldsymbol{\theta}$. We now let $\mathbf{U}$ denote an n-tuple of two-dimensional scalar fields which determine $\hat{\boldsymbol{u}}$. If we choose to represent the strain tensor field as

$$\boldsymbol{E}(\hat{\boldsymbol{u}})(\mathbf{y}, y^3) = E_{ij}(\hat{\boldsymbol{u}})(\mathbf{y}, y^3) \hat{\boldsymbol{a}}^i(\mathbf{y}) \otimes \hat{\boldsymbol{a}}^j(\mathbf{y}) \tag{9}$$

and if we do not model the stretch in the thickness direction, we may write

$$E_{ij}(\hat{\boldsymbol{u}}(\mathbf{U}))(\mathbf{y}, y^3) = \gamma_{ij}(\mathbf{U})(\mathbf{y}) - y^3 \kappa_{ij}(\mathbf{U})(\mathbf{y}), \tag{10}$$

where

$$\begin{aligned}
2\gamma_{\alpha\beta}(\mathbf{U}) &= \hat{\boldsymbol{a}}_\alpha \cdot \partial_\beta \boldsymbol{v} + \partial_\alpha \boldsymbol{v} \cdot \hat{\boldsymbol{a}}_\beta, \\
2\gamma_{\alpha 3}(\mathbf{U}) &= \hat{\boldsymbol{a}}_3 \cdot \partial_\alpha \boldsymbol{v} - \hat{\boldsymbol{a}}_\alpha \cdot \boldsymbol{\beta}, \\
\kappa_{\alpha\alpha}(\mathbf{U}) &= \hat{\boldsymbol{a}}_\alpha \cdot \partial_\alpha \boldsymbol{\beta} - \kappa_\alpha \gamma_{\alpha\alpha}(\mathbf{U}), \\
2\kappa_{12}(\mathbf{U}) &= \hat{\boldsymbol{a}}_1 \cdot \partial_2 \boldsymbol{\beta} + \partial_1 \boldsymbol{\beta} \cdot \hat{\boldsymbol{a}}_2 - \kappa_2 \hat{\boldsymbol{a}}_1 \cdot \partial_2 \boldsymbol{v} - \kappa_1 \partial_1 \boldsymbol{v} \cdot \hat{\boldsymbol{a}}_2, \\
2\kappa_{\alpha 3}(\mathbf{U}) &= \hat{\boldsymbol{a}}_3 \cdot \partial_\alpha \boldsymbol{\beta} - \kappa_\alpha \hat{\boldsymbol{a}}_3 \cdot \partial_\alpha \boldsymbol{v},
\end{aligned} \tag{11}$$

with $\kappa_1 \equiv B_1^1$ and $\kappa_2 \equiv B_2^2$.

Now, if we choose to use the component representations $\boldsymbol{v} = (\boldsymbol{v} \cdot \hat{\boldsymbol{a}}_k)\hat{\boldsymbol{a}}^k$ and $\boldsymbol{\beta} = (\boldsymbol{\beta} \cdot \hat{\boldsymbol{a}}_\alpha)\hat{\boldsymbol{a}}^\alpha$, the equations (11) can be used to obtain the traditional expressions for strains in terms of the covariant derivatives which enable differentiation as (see, for example, [1])

$$\partial_\alpha(v_i \hat{\boldsymbol{a}}^i) = (v_{\beta|\alpha} - B_{\alpha\beta} v_3)\hat{\boldsymbol{a}}^\beta + (v_{3|\alpha} + B_\alpha^\beta v_\beta)\hat{\boldsymbol{a}}^3,$$

with

$$v_{\beta|\alpha} \equiv \partial_\alpha v_\beta - \Gamma_{\alpha\beta}^\nu v_\nu, \quad v_{3|\alpha} \equiv \partial_\alpha v_3. \tag{12}$$

Alternatively, a local orthonormal basis

$$\{\boldsymbol{e}_1(\mathbf{y}), \boldsymbol{e}_2(\mathbf{y}), \boldsymbol{e}_3(\mathbf{y})\} = \{\boldsymbol{e}^1(\mathbf{y}), \boldsymbol{e}^2(\mathbf{y}), \boldsymbol{e}^3(\mathbf{y})\}$$

may be introduced by defining

$$\boldsymbol{e}_\alpha(\mathbf{y}) = \frac{\hat{\boldsymbol{a}}_\alpha(\mathbf{y})}{A_\alpha(\mathbf{y})}, \quad \boldsymbol{e}_3(\mathbf{y}) = \hat{\boldsymbol{a}}_3(\mathbf{y}), \tag{13}$$

with

$$A_\alpha(\mathbf{y}) = |\hat{\boldsymbol{a}}_\alpha(\mathbf{y})| \tag{14}$$

giving the geometric Lamé parameters. One may then choose to use the component representations $\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{e}_k)\boldsymbol{e}^k$ and $\boldsymbol{\beta} = (\boldsymbol{\beta} \cdot \boldsymbol{e}_\alpha)\boldsymbol{e}^\alpha$ in order to compute the strains (11). In this case, the well-known equations for the derivatives of the basis vectors $\boldsymbol{e}_k(\mathbf{y})$ may be applied.

If the orthonormal basis is used, it is also convenient to express the strain tensor field as

$$\boldsymbol{E}(\hat{\boldsymbol{u}})(\mathbf{y}, y^3) = \hat{E}_{ij}(\hat{\boldsymbol{u}})(\mathbf{y}, y^3)\boldsymbol{e}^i(\mathbf{y}) \otimes \boldsymbol{e}^j(\mathbf{y}), \tag{15}$$

with the different components of the versions (9) and (15) related by

$$\hat{E}_{\alpha\beta}(\hat{\boldsymbol{u}}) = \frac{E_{\alpha\beta}(\hat{\boldsymbol{u}})}{A_\alpha A_\beta}, \quad \hat{E}_{\alpha 3}(\hat{\boldsymbol{u}}) = \frac{E_{\alpha 3}(\hat{\boldsymbol{u}})}{A_\alpha}, \quad \hat{E}_{33}(\hat{\boldsymbol{u}}) = E_{33}(\hat{\boldsymbol{u}}). \tag{16}$$

This alternative can be followed to create the component expressions of strains so that their geometric parameters depend only on A_α. This choice is taken as the starting point in the study [9].

We finally note that here a basic material model of linear elasticity is assumed, so that the strain energy density is based on the relations [4]

$$\begin{aligned}
W(\boldsymbol{E}) &= \frac{1}{2} T^{km} E_{mk} \\
T^{rk} &= \lambda(A^{qp} E_{pq}) A^{rk} + 2\mu A^{mr} A^{ks} E_{ms}
\end{aligned}$$

where the stress tensor T^{rk} is expressed in terms of the Lamé parameters λ and μ of the material. In the case of an ideal surface parametrization the application of the material law can thus be made without any reference to the metric tensor.

Shell equations by surface reparametrization

Constant-coefficient approximations of the strains over a local domain can be obtained in a straightforward manner by using the differential geometric results derived in [3]. The key idea employed therein is that the local graph of the mid-surface

$$\boldsymbol{\varphi}_S(\mathbf{x}) \equiv (\mathbf{x}, z(\mathbf{x})), \tag{17}$$

with $\mathbf{x}$ denoting a point on a planar region S, may be reparametrized by a composition

$$\boldsymbol{\varphi}_S \circ \boldsymbol{\phi} : \omega \subset \mathbb{R}^2 \to \mathbb{E}^3 \qquad (\text{with } S = \boldsymbol{\phi}(\omega)) \tag{18}$$

which can be thought of as consisting of polynomial perturbations added to the simplest reparametrization functions (that is, $x^\alpha(\mathbf{y}) = y^\alpha$), provided we have adjusted the orientation of the coordinate axes to agree with the principal directions of the mid-surface at the origin $\mathbf{o}$. The coefficients of the perturbation monomials are specified such that the resulting surface basis vectors are orthogonal and the components of the second fundamental form of the surface remain diagonal up to a desired degree as we move away from the origin. In practice the Taylor polynomial

$$\begin{aligned} z(x^1, x^2) =& 1/2a(x^1)^2 + 1/2b(x^2)^2 + 1/6d(x^1)^3 + 1/2e(x^1)^2x^2 + \\ & 1/2fx^1(x^2)^2 + 1/6g(x^2)^3 + O(|\mathbf{x} - \mathbf{o}|^4) \end{aligned} \tag{19}$$

is employed to analyse how well the desired conditions are respected.

The task of the reparametrization is thus to find the coefficients occurring in the choice

$$\begin{aligned} \boldsymbol{\phi}(\mathbf{y}) \equiv (y^1 &+ \frac{1}{2}c_1(y^2)^2 + c_2y^1y^2 + \frac{1}{2}c_3(y^1)^2 + \\ & \frac{1}{6}c_4(y^1)^3 + \frac{1}{2}c_5(y^1)^2y^2 + \frac{1}{2}c_6y^1(y^2)^2 + \frac{1}{6}c_7(y^2)^3, \\ y^2 &+ \frac{1}{2}b_1(y^1)^2 + b_2y^1y^2 + \frac{1}{2}b_3(y^2)^2 + \\ & \frac{1}{6}b_4(y^1)^3 + \frac{1}{2}b_5(y^1)^2y^2 + \frac{1}{2}b_6y^1(y^2)^2 + \frac{1}{6}b_7(y^2)^3). \end{aligned} \tag{20}$$

It should be noted that the case of $a = b$ (that of an umbilical point) is problematic when the shell is not spherical, since in this case we can diagonalize the second fundamental form only if $e = 0$ and $f = 0$. In view of the aim of the present discussion, it is however sufficient to consider the case $a \neq b$. For a discussion of possible options in the case $a = b$, we only refer to [3].

The full expansions of the differential quantities of our interest here can be found in [3]. For the purpose of the present study, it is sufficient to truncate these expansions as

$$\begin{aligned} A_{11}(\mathbf{y}) = A_{22}(\mathbf{y}) &= 1 - 2c_1y^1 + 2c_2y^2 + O(|\mathbf{y} - \mathbf{o}|^2), \\ \frac{1}{R_1}(\mathbf{y}) &= a + dy^1 + ey^2 + O(|\mathbf{y} - \mathbf{o}|^2), \\ \frac{1}{R_2}(\mathbf{y}) &= b + fy^1 + gy^2 + O(|\mathbf{y} - \mathbf{o}|^2), \\ \Gamma^1_{11}(\mathbf{y}) = \Gamma^2_{12}(\mathbf{y}) = -\Gamma^1_{22}(\mathbf{y}) &= -c_1 + O(|\mathbf{y} - \mathbf{o}|), \\ \Gamma^2_{22}(\mathbf{y}) = \Gamma^1_{12}(\mathbf{y}) = -\Gamma^2_{11}(\mathbf{y}) &= c_2 + O(|\mathbf{y} - \mathbf{o}|), \end{aligned} \tag{21}$$

where

$$c_1 = \frac{f}{b-a} \quad \text{and} \quad c_2 = \frac{e}{b-a}. \tag{22}$$

We note that the conditions (22) result from the requirement of the diagonalization of the second fundamental form. It is also noted that the perturbation terms of (20) up to order 2 in y^α can be expressed solely in terms of these two constants as

$$\begin{aligned} x^1 &= y^1 + \frac{1}{2}c_1(y^2)^2 + c_2 y^1 y^2 - \frac{1}{2}c_1(y^1)^2, \\ x^2 &= y^2 - \frac{1}{2}c_2(y^1)^2 - c_1 y^1 y^2 + \frac{1}{2}c_2(y^2)^2. \end{aligned} \tag{23}$$

For later use, we note that the first equation of (21) gives

$$\frac{\partial A_1}{\partial y^2}(\mathbf{0}) = c_2, \quad \frac{\partial A_2}{\partial y^1}(\mathbf{0}) = -c_1. \tag{24}$$

In addition, we obtain

$$\begin{aligned} c_{11} &= \frac{\partial}{\partial y^1}(\frac{1}{R_1})(\mathbf{0}) = d, \\ c_{12} &= \frac{\partial}{\partial y^2}(\frac{1}{R_1})(\mathbf{0}) = e, \\ c_{21} &= \frac{\partial}{\partial y^1}(\frac{1}{R_2})(\mathbf{0}) = f, \\ c_{22} &= \frac{\partial}{\partial y^2}(\frac{1}{R_2})(\mathbf{0}) = g. \end{aligned} \tag{25}$$

Our reparametrization is then seen to respect the Codazzi conditions evaluated at $\mathbf{y} = \mathbf{0}$ as

$$\frac{\partial A_1}{\partial y^2}(\frac{1}{R_2} - \frac{1}{R_1})\Big|_{\mathbf{y}=\mathbf{0}} = A_1(\mathbf{0})c_{12}, \quad \frac{\partial A_2}{\partial y^1}(\frac{1}{R_1} - \frac{1}{R_2})\Big|_{\mathbf{y}=\mathbf{0}} = A_2(\mathbf{0})c_{21}. \tag{26}$$

It is now an easy task to obtain constant-coefficient expressions for strains by taking into account only the leading terms. For example, the membrane strains with respect to the orthonormal basis may be approximated as

$$\begin{aligned} \hat{\gamma}_{11} &= \frac{1}{A_1}\frac{\partial v_1}{\partial y^1} + \frac{v_2}{A_1 A_2}\frac{\partial A_1}{\partial y^2} + \frac{v_3}{R_1} \approx \frac{\partial v_1}{\partial y^1} + c_2 v_2 + a v_3, \\ \hat{\gamma}_{22} &= \frac{1}{A_2}\frac{\partial v_2}{\partial y^2} + \frac{v_1}{A_1 A_2}\frac{\partial A_2}{\partial y^1} + \frac{v_3}{R_2} \approx \frac{\partial v_2}{\partial y^2} - c_1 v_1 + b v_3, \\ 2\hat{\gamma}_{12} &= \frac{1}{A_1}\frac{\partial v_2}{\partial y^1} + \frac{1}{A_2}\frac{\partial v_1}{\partial y^2} - \frac{v_1}{A_1 A_2}\frac{\partial A_1}{\partial y^2} - \frac{v_2}{A_1 A_2}\frac{\partial A_2}{\partial y^1} \approx \frac{\partial v_2}{\partial y^1} + \frac{\partial v_1}{\partial y^2} - c_2 v_1 + c_1 v_2, \end{aligned} \tag{27}$$

where the components $v_k = \boldsymbol{v} \cdot \boldsymbol{e}_k$ are used. We thus conclude that we need four geometric parameters

$$a, \; b, \; c_1 = \frac{f}{b-a}, \; c_2 = \frac{e}{b-a}$$

to obtain a constant-coefficient form of the strain energy density.

An alternative derivation of shallow shell equations

The constant-coefficient formulation of shell equations (as we here consider only the expressions for the membrane strains, we follow the abbreviations of the original study and refer to it as the $\mathrm{LNK_S}$ formulation below) given by Pitkäranta, Matache and Schwab in [9] assumes that a global surface parametrization by lines of curvature coordinates is available. Now this parametrization is subject to an additional constraint that the curvilinear coordinates in the surface correspond to arc-length parameters $\boldsymbol{\alpha} = (\alpha^1, \alpha^2)$ such that $A_1(\alpha^1, 0) = 1$ and $A_2(0, \alpha^2) = 1$. The assumption of shallowness is also made, so that $\epsilon = L/R \ll 1$, where L characterizes the maximal distance of two points along a part of surface considered and $1/R$ measures the curvature.

The derivation of constant-coefficient expressions for the strains is then done in terms of eight pointwise parameters (cf. (24) and (25))

$$k_\alpha = \frac{1}{R_\alpha}(\mathbf{0}),\ a_1 = \frac{\partial A_1}{\partial \alpha^2}(\mathbf{0}),\ a_2 = \frac{\partial A_2}{\partial \alpha^1}(\mathbf{0}),\ c_{\alpha\beta} = \frac{\partial}{\partial \alpha^\beta}(\frac{1}{R_\alpha})(\mathbf{0}), \tag{28}$$

but the final formulation depends only on two curvature parameters k_α which now satisfy $k_1 = a$ and $k_2 = b$. This is in contrast to the approximation of the preceding section which needs four geometric parameters. Our aim here is to investigate from where this quite substantial difference may emerge.

Although the original derivation of the $\mathrm{LNK_S}$ formulation does not utilize the notion of graph as given by (17), its first step can be constructed by considering again the reparametrization of (17). The original study assumes that the nonparametric form (17) is related to lines of curvature coordinates by a parametric representation

$$(x^1(\boldsymbol{\alpha}), x^2(\boldsymbol{\alpha}), z(x^1(\boldsymbol{\alpha}), x^2(\boldsymbol{\alpha})))$$

with the approximations

$$\begin{aligned} x^1(\boldsymbol{\alpha}) &= \alpha^1 + a_1\alpha^1\alpha^2 - \frac{a_2}{2}(\alpha^2)^2, \\ x^2(\boldsymbol{\alpha}) &= \alpha^2 + a_2\alpha^1\alpha^2 - \frac{a_1}{2}(\alpha^1)^2. \end{aligned} \tag{29}$$

In order to make the connection between (29) and the reparametrization of the preceding section fully explicit, we may first calculate the arc-length parameters as

$$s^1 = \int_0^{y^1} \sqrt{A_{11}(t,0)}dt = y^1 - \frac{1}{2}c_1(y^1)^2 + o[(y^1)^2], \tag{30}$$

$$s^2 = \int_0^{y^2} \sqrt{A_{22}(0,t)}dt = y^2 + \frac{1}{2}c_2(y^2)^2 + o[(y^2)^2]. \tag{31}$$

By using these, the truncated expressions (23) can be transformed into the form

$$\begin{aligned} x^1(s^1,s^2) &= s^1 + \frac{1}{2}c_1(s^2)^2 + c_2s^1s^2, \\ x^2(s^1,s^2) &= s^2 - \frac{1}{2}c_2(s^1)^2 - c_1s^1s^2, \end{aligned} \tag{32}$$

so this parametrization is of the same form as the transformation (29) used in the derivation of the $\mathrm{LNK_S}$ formulation. Finally, if we evaluate the parameters a_α for (32), we conclude that they satisfy

$$a_1 = c_2, \quad a_2 = -c_1.$$

Interestingly, these relations give the interpretation for a_α in terms of the coefficients of the Taylor polynomial of degree 3.

Despite the agreement of the two parametrizations of the graph up to the terms of order 2, the constant-coefficient strains of the $\mathrm{LNK_S}$ model as given in [9] do not agree with those obtained in the preceding section. At first sight, the situation might seem puzzling as the above considerations which led us to the approximations (27) do not in general suggest a further means to get rid of terms of the type

$$v_\nu \frac{\partial A_\alpha}{\partial y^\beta}. \tag{33}$$

Indeed, the present author believes that these expressions might be the simplest consistent approximations when lines of curvature coordinates are approximated by means of the full parametrization of the form (20).

However, the derivation presented in [9] eventually discards the parametric form of the graph based on the transformation (32) by means of transformation to a nonparametric form where the Cartesian coordinates of the planar region S are directly employed as curvilinear coordinates, i.e., the curvilinear coordinates are defined via the choice $x^\alpha(\mathbf{y}) = y^\alpha$. This naturally gives rise to a different metric tensor which is seen to conform to the theory of locally Cartesian coordinates [5]. Therefore, the first partial derivatives of the components of the metric tensor (as wells as the Christoffel symbols) evaluated at the origin are zero which enables the derivation of approximate strain expressions depending only on the curvature parameters (cf. [2, Section 2]). We however mention that using the nonparametric representation together with the Taylor polynomial of order 3 does not lead to an optimally accurate diagonalization of the components of the fundamental forms. Therefore, if more accurate expressions for strains were sought, this representation would lead to adopting a broader repertoire of tensor computations, while the use of the full parametric form of the graph would even then offer the simplicity related to the lines of curvature coordinates.

Finally we note that if the approximate strains given by (27) are applied, it may of course happen that their dependence on the two non-curvature parameters c_α become insignificant. Therefore, let us consider under what conditions the terms of the type (33) become negligible if we utilize the reparametrization of the preceding section. Since the expansions (21) give

$$e = \frac{\partial}{\partial y^2}\left(\frac{1}{R_1}\right)(\mathbf{0}), \quad f = \frac{\partial}{\partial y^1}\left(\frac{1}{R_2}\right)(\mathbf{0}),$$

the conditions $|(\partial A_1/\partial y^2)(\mathbf{0})| = |e/(b-a)| \ll 1$ and $|(\partial A_2/\partial y^1)(\mathbf{0})| = |f/(b-a)| \ll 1$ lead us to the requirement

$$\left|\frac{\partial}{\partial y^\beta}\left(\frac{1}{R_\alpha}\right)\right| \div \left|\frac{1}{R_2} - \frac{1}{R_1}\right| \ll 1. \tag{34}$$

That is, in the case of the parametrization considered here, slowly varying curvatures are required. This condition is unrelated to the assumption of shallowness, which ensures that $|\mathbf{y}-\mathbf{o}|/|R| \ll 1$, or $|\mathbf{y}-\mathbf{o}| \cdot \sqrt{a^2+b^2} \ll 1$, with $\mathbf{y} \in \omega$. We note that a basic example of the case which allows for the additional reduction is the case of a straight cylindrical shell, as its curvature is a constant and then (34) naturally holds.

Table 1. The patchwise approximations of geometric parameters.

	S_1	S_2	S_3	S_4
a	$1.90 \cdot 10^{-1}$	$1.84 \cdot 10^{-1}$	$1.82 \cdot 10^{-1}$	$1.76 \cdot 10^{-1}$
b	$-1.04 \cdot 10^{-0}$	$-5.76 \cdot 10^{-1}$	$-7.49 \cdot 10^{-1}$	$-4.89 \cdot 10^{-1}$
d	$1.00 \cdot 10^{-2}$	$6.44 \cdot 10^{-2}$	$5.97 \cdot 10^{-2}$	$4.64 \cdot 10^{-2}$
e	$8.08 \cdot 10^{-2}$	$1.89 \cdot 10^{-2}$	$3.64 \cdot 10^{-7}$	$-6.55 \cdot 10^{-3}$
f	$-2.89 \cdot 10^{-0}$	$-1.10 \cdot 10^{-0}$	$-1.58 \cdot 10^{-0}$	$-8.55 \cdot 10^{-1}$
g	$2.28 \cdot 10^{-1}$	$1.53 \cdot 10^{-2}$	$-3.48 \cdot 10^{-7}$	$3.12 \cdot 10^{-2}$
c_1	$2.35 \cdot 10^{-0}$	$1.45 \cdot 10^{-0}$	$1.70 \cdot 10^{-0}$	$1.29 \cdot 10^{-0}$
c_2	$-6.57 \cdot 10^{-2}$	$-2.49 \cdot 10^{-2}$	$3.91 \cdot 10^{-7}$	$9.85 \cdot 10^{-3}$

A computational example

We shall next illustrate that in practical applications the non-curvature parameters c_1 and c_2 can be significant as compared with the curvature parameters, so that evidence is given that the strain expressions of (27) cannot indeed be simplified further. To illustrate this, we now consider a shell with the mid-surface

$$S = [1/4, 3/4] \times [0, 1/2] \ni (u, v) \mapsto (u, v, \exp(-\sqrt{(u^2 + v^2)/3})$$

and divide S regularly into four patches S_i. In practice a mesh of four finite elements is generated. The geometric parameters to generate the reparametrization of each patch by approximate lines of curvature coordinates have already been computed in [3] and are also reproduced in Table 1, together with the values of the constants c_i. Here we have $c_1 = O(1)$, so we see no consistent way to express the constant-coefficient approximations given by (27) in terms of only two geometric parameters a and b.

Concluding remarks

We have here shown that the application of the method described in [3] to produce a consistently accurate reparametrization by lines of curvature coordinates leads to the simplified expressions of strains which depend on four geometric parameters. This is in contrast to the earlier derivations which need only two parameters of curvature. Such discrepancy is intimately related to the properties of the metric tensors that are attainable by means of different parametrizations of a given surface.

While the reparametrization method applied here produces a more faithful representation of lines of curvature as compared with the earlier formulations, it does not lead to the metric tensor that allows the curvilinear coordinates to be locally Cartesian [5]. Therefore the associated Christoffel symbols do not in general vanish at a preassigned point. On the other hand, the two-parameter expressions of related studies have been reached by using locally Cartesian curvilinear coordinates at the cost of having greater errors in the diagonalization conditions which are natural for the fundamental forms corresponding to lines of curvature coordinates. Such errors are insignificant in the case of rudimentary models, but if more accurate ways to evaluate the strain energy density were sought, the presence of non-diagonal components of the fundamental forms would lead to adopting a broader repertoire of tensor computations, whereas the consistently accurate reparametrization by lines of curvature coordinates is designed to avoid such additional complexity of computation.

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